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Some Weighted Polynomial Inequalities in L^2-Norm

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In this paper we give a new characterization of the classical orthogonal polynomials (Hermite, generalized Laguerre, Jacobi) by extremal properties in some weighted polynomial inequalities in L^2-norm.

1. INTRODUCTION

Let $\mathscr{P}_n$ be the set of all algebraic polynomials of degree at most n on an interval (a, b). Markov [11] proved in 1889 the following

THEOREM 1.1. *Let* $(a, b) = [-1, 1]$ *and* $\|f\|_{\infty,[-1,1]} = \sup_{t \in [-1,1]} |f(t)|$. *Then*

$$\|P'\|_{\infty,[-1,1]} \leq n^2 \|P\|_{\infty,[-1,1]} \tag{1}$$

for each $P \in \mathscr{P}_n$.

This result was followed by several extensions by other authors both from bounded to unbounded intervals, also from L^∞ norm to other norms, especially norms involving weight functions (cf. Milovanović [13]). In the latter case, it is common to talk about "weighted polynomials" (instead of "weighted norms"). We will use this denomination throughout this paper.

For weighted polynomials on unbounded intervals, a first result was obtained by Milne [12] in 1931. A transformed form of his result can be described as follows:

THEOREM 1.2. *Let* $(a, b) = (-\infty, +\infty)$ *and* $\|f\|_{\infty,\mathbf{R}} = \sup_{t \in \mathbf{R}} |f(t)|$. *There exists a positive constant C such that*

$$\left\| [\exp(-t^2/2)P]' \right\|_{\infty,\mathbf{R}} \leq C\sqrt{n} \left\| \exp(-t^2/2)P \right\|_{\infty,\mathbf{R}} \tag{2}$$

for each $P \in \mathscr{P}_n$.

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Using the L^r-norm on $(-\infty, +\infty)$, given by

$$\|f\|_r = \left(\int_{\mathbf{R}} |f(t)|^r \, dt\right)^{1/r}, \qquad 0 \le r < +\infty,$$
$$= \operatorname*{ess\,sup}_{t \in \mathbf{R}} |f(t)|, \qquad r = +\infty,$$

Freud [6] proved:

THEOREM 1.3. *Let $P \in \mathscr{P}_n$ and $1 \le r \le \infty$. Then there exists a positive constant C_1, independent of n, such that*

$$\|\exp(-t^2/2)P'\|_r \le C_1 n^{1/2} \|\exp(-t^2/2)P\|_r. \tag{3}$$

In 1983, a generalization of inequality (2) to L^r-norms for higher derivatives was given by Zalik [17]. He proved the following:

THEOREM 1.4. *Given $s > 0$, $1 \le r \le \infty$. Then, there exists a positive constant D_1, independent of n, such that*

$$\left\|\left[\exp(-t^2/2)P\right]^{(s)}\right\|_r \le D_1 n^{s/2} \|\exp(-t^2/2)P\|_r, \tag{4}$$

for every $P \in \mathscr{P}_n$.

In proving this assertion, he used the following result:

THEOREM 1.5. *Given $s > 0$, $0 < q < \infty$ and $1 \le r \le \infty$. Then there exists a positive constant D_2, independent of n, such that*

$$\||t|^q \exp(-t^2/2)P\|_r \le D_2 n^{q/2} \|\exp(-t^2/2)P\|_r \tag{5}$$

for every $P \in \mathscr{P}_n$.

The above inequalities are optimal in the sense that constants D_1 and D_2 cannot be replaced by sequences that converge to zero as $n \to \infty$.

Similar weighted polynomial inequalities on the interval $(0, +\infty)$ were considered by Zalik [18].

An elegant proof of Freud's inequality (3) was found by Nevai and Freud [14]. The idea has been applied to a wide class of Freud-type weights

$$w(t) = \exp(-Q(t)), \qquad t \in \mathbf{R},$$

with some suitable conditions on the function $t \mapsto Q(t)$. This simplification of Freud's proof was subsequently applied by Levin and Lubinsky [9], Lubinsky and Mthembu [10], and others.

The classical orthogonal polynomials can be specified as:

the Hermite polynomials $H_n(t)$, orthogonal on the interval $(-\infty, +\infty)$ with respect to the measure $\exp(-t^2)\,dt$;

the generalized Laguerre polynomials $L_n^{(s)}(t)$, orthogonal respect to the measure $t^s \exp(-t)\,dt (s > -1)$ on the interval $(0, +\infty)$;

the Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$ which are orthogonal on the interval $[-1, 1]$ with respect to the measure $(1-t)^\alpha(1+t)^\beta\,dt (\alpha, \beta > -1)$.

Let $d\sigma(t)$ be the measure of the classical orthogonal polynomials, i.e., $d\sigma(t) = w(t)\,dt$ on (a, b), with

$$(a,b) = \begin{cases} (-\infty, +\infty), & \text{Hermite case,} \\ (0, +\infty), & \text{generalized Laguerre case,} \\ [-1, 1], & \text{Jacobi case.} \end{cases}$$

The weight $t \mapsto w(t)$ satisfying the differential equation of the first order

$$\frac{d}{dt}(A(t)w(t)) = B(t)w(t),$$

where the function $t \mapsto A(t)$ is given by

$$A(t) = \begin{cases} 1, & \text{Hermite case,} \\ t, & \text{generalized Laguerre case,} \\ 1-t^2, & \text{Jacobi case,} \end{cases} \tag{6}$$

and $t \mapsto B(t)$ is a polynomial of degree at most one (cf. Szegő [16]).

In this paper we give a new characterization of the classical orthogonal polynomials by extremal properties in some weighted polynomial inequalities. More precisely, we prove the following:

THEOREM 1.6. *Let $(\mathscr{P}_n, \|\cdot\|_{d\sigma})$ be the class of algebraic polynomials of degree at most n equipped with the norm $\|P\|_{d\sigma} = (\int_{\mathbf{R}} |P(t)|^2\,d\sigma(t))^{1/2}$, where $d\sigma(t)$ is the measure of the classical orthogonal polynomials. Let w be the weight function corresponding to $d\sigma(t)$. Then for all $P \in \mathscr{P}_n$ we have*

$$\left\|\frac{1}{\sqrt{w}}\frac{d}{dt}(V(t)P)\right\|_{d\sigma}^2 + \left\|\sqrt{A(t)C(t)}\,P\right\|_{d\sigma}^2 \le \beta_n \|P\|_{d\sigma}^2, \tag{7}$$

where $V(t) = \sqrt{A(t)w(t)}$, *and*

$$C(t) = \begin{cases} t^2, & \textit{Hermite case}, \\ \dfrac{1}{4}\left(\dfrac{\alpha^2 - 1}{t^2} + 1\right), & \textit{generalized Laguerre case}, \\ \dfrac{1}{4}\left(\dfrac{\alpha^2 - 1}{(1-t)^2} + \dfrac{\beta^2 - 1}{(1+t)^2}\right), & \textit{Jacobi case}, \end{cases}$$

and

$$\beta_n = \begin{cases} 2n + 1, & \textit{Hermite case}, \\ n + (\alpha + 1)/2, & \textit{generalized Laguerre case}, \\ n(n + \alpha + \beta + 1) & \textit{Jacobi case}. \\ \quad +(\alpha + 1)(\beta + 1)/2, & \end{cases}$$

The supremum is attained if only if $P(t) = \gamma\Pi_{n,w}(t)$, *where* $\Pi_{n,w}$ *is the polynomial of degree n which is orthogonal to* $\mathscr{P}_{n-1}$, *with respect to the weight function* $t \mapsto w(t)$, *and* $\gamma(\neq 0)$ *is an arbitrary real constant.*

Similar characterizations of these polynomials are given in Agarwal and Milovanović [2], Guessab and Milovanović [4, 5], and Varma [15].

2. Proof of Theorem 1.6

The proof of theorem is based on a certain transformation of homogeneous linear differential equations of the second order, satisfied by the classical orthogonal polynomials (cf. Szegő [16])

$$\frac{d}{dt}\left(A(t)w(t)\frac{dy}{dt}\right) + \lambda_n w(t)y = 0, \tag{8}$$

where the spectral parameter

$$\lambda_n = \begin{cases} 2n, & \text{Hermite case}, \\ n, & \text{generalized Laguerre case}, \\ n(n + \alpha + \beta + 1), & \text{Jacobi case}, \end{cases}$$

and the function $t \mapsto A(t)$ is given by (6). If in (8) we introduce $U(t) = V(t)y$, where

$$V(t) = \sqrt{A(t)w(t)},$$

we obtain the following important transformed equation:

$$\frac{d^2}{dt^2}U - C(t)U + \beta_n \frac{U}{A(t)} = 0. \tag{9}$$

$C(t)$ and β_n are defined in Theorem 1.6. Now, let $d\sigma(t)$ be the measure of the classical orthogonal polynomials, for $P \in \mathscr{P}_n$, denote by $\|P\|_{d\sigma} = \sqrt{(P,P)_{d\sigma}}$ where

$$(f,g)_{d\sigma} = \int_{\mathbf{R}} f(t)g(t)\,d\sigma(t) \qquad (f,g \in L^2(\mathbf{R})).$$

Let $P \in \mathscr{P}_n$. Then we can write $P(t) = \sum_{\nu=0}^{n} a_\nu \Pi_{\nu,w}(t)$, where the coefficients a_k are uniquely determined and $\Pi_{\nu,w}$ is the orthogonal polynomial of degree ν which is orthogonal to $\mathscr{P}_{\nu-1}$, with respect to the weight function $t \mapsto w(t)$. Now, we define

$$\mathscr{L}(P) = \frac{d^2}{dt^2}(V(t)P) - C(t)V(t)P + \beta_n \frac{V(t)P}{A(t)}.$$

Then we have from (9)

$$\begin{aligned}\mathscr{L}(P) &= \sum_{\nu=0}^{n} a_\nu \mathscr{L}(\Pi_{\nu,w}) \\ &= \sum_{\nu=0}^{n} a_\nu(\beta_n - \beta_\nu)\frac{V(t)}{A(t)}\Pi_{\nu,w}.\end{aligned}$$

It follows that

$$\begin{aligned}(w^{-1}\mathscr{L}(P), V(t)P)_{d\sigma} &= \sum_{\nu,\mu=0}^{n} (\beta_n - \beta_\nu)a_\nu a_\mu(\Pi_{\nu,w}, \Pi_{\mu,w})_{d\sigma} \\ &= \sum_{\nu=0}^{n} (\beta_n - \beta_\nu)a_\nu^2\|\Pi_{\nu,w}\|_{d\sigma}^2.\end{aligned}$$

However,

$$\begin{aligned}(w^{-1}\mathscr{L}(P), V(t)P)_{d\sigma} = &\left(w^{-1}\frac{d^2}{dt^2}(V(t)P), V(t)P\right)_{d\sigma} \\ &- \left\|\sqrt{A(t)C(t)}\,P\right\|_{d\sigma}^2 + \beta_n\|P\|_{d\sigma}^2.\end{aligned}$$

Integrating by parts, we have

$$\left(w^{-1}\frac{d^2}{dt^2}(V(t)P),V(t)P\right)_{d\sigma} = -\left\|\frac{1}{\sqrt{w}}\frac{d}{dt}(V(t)P)\right\|_{d\sigma}^2.$$

Then, we find

$$(w^{-1}\mathscr{L}(P),V(t)P)_{d\sigma} = -\left\|\frac{1}{\sqrt{w}}\frac{d}{dt}(V(t)P)\right\|_{d\sigma}^2 - \left\|\sqrt{A(t)C(t)}\,P\right\|_{d\sigma}^2 + \beta_n\|P\|_{d\sigma}^2.$$

Since $(w^{-1}\mathscr{L}(P), V(t)P)_{d\sigma} \geq 0$, we obtain

$$\left\|w^{-1}\frac{d}{dt}(V(t)P)\right\|_{d\sigma}^2 + \left\|\sqrt{A(t)C(t)}\,P\right\|_{d\sigma}^2 \leq \beta_n\|P\|_{d\sigma}^2.$$

Finally, we see that

$$(w^{-1}\mathscr{L}(P),V(t)P)_{d\sigma} = \sum_{\nu=0}^{n}(\beta_n - \beta_\nu)a_\nu^2\|\Pi_{\nu,w}\|_{d\sigma}^2 = 0,$$

if and only if $a_k = 0$, $\forall k = 0,1,\ldots,n-1$, and $a_n \in \mathbf{R}$; i.e. $P = c\Pi_n$, where $c \in \mathbf{R}$. This completes the proof of Theorem 1.6.

3. Special Cases

In this section we will give examples with the measures of the classical orthogonal polynomials.

In the Hermite case, Theorem 1.6 reduces to the following:

Corollary 3.1. *If P is a real polynomial of degree n, then we have*

$$\left\|\exp(t^2/2)\left[\exp(-t^2/2)P\right]'\right\|_2^2 + \|tP\|_2^2 \leq (2n+1)\|P\|_2^2,$$

where $\|f\|_2^2 = \int_{-\infty}^{+\infty}\exp(-t^2)f(t)^2\,dt$. *The supremum is attained only if $P(t) = \gamma H_n(t)$, where γ is an arbitrary real constant.*

Remark 3.1. Note that the left hand side of the above inequality is the sum of the left handside of the inequality considered in Theorem 1.4 and the one in Theorem 1.5, for $s = 1$, $r = 2$, and $q = 1$. The number $2n + 1$

appearing in the right hand side of the above formula is optimal. Therefore, we obtained the best estimate possible for $(D_1^2 + D_2^2)n$, where D_1 and D_2 are constants given in Theorem 1.4 and Theorem 1.5.

For the weight function associated with the generalized Laguerre polynomials we have:

COROLLARY 3.2. *Let* $w_L(t) = t^{(s+1)/2}\exp(-t/2)$ $(s > 1)$ *on* $(0, +\infty)$. *Let P be a given real polynomial of degree n. Then we have*

$$\|t^{-s/2}\exp(t/2)[w_L P]'\|_2^2 + \frac{1}{4}\left\|\left(\frac{s^2-1}{t} + t\right)^{1/2} P\right\|_2^2 \leq \beta_n \|P\|_2^2,$$

where $\beta_n = n + (s+1)/2$ *and* $\|f\|_2^2 = \int_0^\infty t^s \exp(-t) f(t)^2\,dt$. *The supremum is attained only if* $P(t) = \gamma L_n^s(t)$, *where* γ *is an arbitrary real constant.*

Finally, in the Jacobi case, we get:

COROLLARY 3.3. *Let* $w_J(t) = (1-t)^{(\alpha+1)/2}(1+t)^{(\beta+1)/2}$ $(\alpha, \beta > 1)$. *Let P be a given real polynomial of degree n. Then we have*

$$\|(1-t)^{-\alpha/2}(1+t)^{-\beta/2}[w_J P]'\|_2^2 + \frac{1}{4}\|\sqrt{1-t^2}\,H(t)P\|_2^2 \leq \lambda_n \|P\|_2^2,$$

where

$$H(t) = \left(\frac{\alpha^2-1}{(1-t)^2} + \frac{\beta^2-1}{(1+t)^2}\right)^{1/2},$$

$$\beta_n = n(n+\alpha+\beta+1) + (\alpha+1)(\beta+1)/2,$$

and $\|f\|_2^2 = \int_{-1}^1 (1-t)^\alpha (1+t)^\beta f(t)^2\,dt$. *The supremum is attained only if* $P(t) = \gamma P_n^{(\alpha,\beta)}(t)$, *where* γ *is an arbitrary real constant.*

Remark 3.2. In the case $\alpha = \beta = 1$, we have for all $P \in \mathscr{P}_n$ the inequality

$$\int_{-1}^1 [(1-t^2)P(t)]'^2\,dt \leq (n+1)(n+2)\int_{-1}^1 (1-t^2)P(t)^2\,dt$$

or in an equivalent form. If $Q \in \mathscr{P}_{n+2}$ is such that $Q(\pm 1) = 0$, then

$$\int_{-1}^1 Q'(t)^2\,dt \leq (n+1)(n+2)\int_{-1}^1 \frac{Q^2(t)}{1-t^2}\,dt. \tag{10}$$

Remark 3.3. Daugavet and Rafal'son [3] and Konjagin [8] considered the extremal problems of the form

$$\|P^{(m)}\|_{p,\nu} \leq A_{n,m}(r,\mu;p,\nu)\|P\|_{r,\mu} \qquad (P \in \mathscr{P}_n), \tag{11}$$

where

$$\|f\|_{r,\nu} = \left(\int_{-1}^{1} \left|f(t)(1-t^2)^{\mu}\right|^r dt\right)^{1/r}, \qquad 0 \leq r < +\infty,$$

$$= \operatorname*{ess\,sup}_{-1 \leq t \leq 1} |f(t)|(1-t^2)^{\mu}, \qquad r = +\infty.$$

The case when $p = r \geq 1$, $\mu = \nu = 0$, and $m = 1$, was considered by Hille *et al.* [7]. The exact constant $A_{n,m}(r,\mu;p,\nu)$ is known in a few cases, for example, $A_{n,1}(\infty,0;\infty,0) = n^2$ is the best constant in Markov's inequality (1), and $A_{n,1}(\infty,0;\infty,1/2) = n$ is the best constant in Bernstein's inequality [1]. Also, we have

$$A_{n,m}(2,\mu;2,\mu+m/2) = \sqrt{\frac{n!\Gamma(n+4\mu+m+1)}{(n-m)!\Gamma(n+4\mu+1)}}.$$

The inequality (10) can be represented in the form

$$\|Q'\|_{2,0} \leq A_{n,1}(2,0;2,-1/2)\|Q\|_{2,-1/2}.$$

This formula extends formula (11) to a case when the weight function has a nonintegrable singularity.

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